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Existence and uniqueness of solutions for multi-term nonlinear fractional integro-differential equations

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Abstract

In this manuscript, by using the fixed point theorems, the existence and the uniqueness of solutions for multi-term nonlinear fractional integro-differential equations are reported. Two examples are presented to illustrate our results.

Keywords: Caputo fractional derivative; fixed point theorem; multi-term nonlinear fractional differential equation

1 Introduction

The study of fractional differential equations ranges from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions. Fractional differential equations appear naturally in a number of fields such as physics, polymer rheology, regular variational in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, *etc.* An excellent account in the study of fractional differential equations can be found in [1, 2] and [3]. For more details and examples, one can study [4–13] and [14]. It is considerable that there are many works about fractional integro-differential equations (see, for example, [15–18] and [19]).

In 2007, Xinwei and Landong reviewed the existence of solutions for the nonlinear fractional differential equation

$$^cD^\alpha u(t) = f\big(t, u(t), {}^cD^\beta u(t)\big) \quad (0 < t < 1)$$

with boundary values $u(0) = u'(1) = 0$ or $u'(0) = u(1) = 0$ or $u(0) = u(1) = 0$, where $1 < \alpha \le 2$, $0 < \beta \le 1$, and f is continuous on $[0,1] \times \mathbb{R} \times \mathbb{R}$ [20]. In 2009, Su and Zhang studied the existence and uniqueness of solutions for the following nonlinear two-point fractional boundary value problem

$$^cD^\alpha u(t) = f\big(t, u(t), {}^cD^\beta u(t)\big) \quad (0 < t < 1)$$

with boundary values $a_1u(0) - a_2u'(0) = A$ and $b_1u(1) + b_2u'(1) = B$, where α, β, a_i, b_i $(i = 1, 2)$ satisfy certain conditions [21]. In 2010, Ahmad and Sivasundaram studied the existence of solutions for the nonlinear fractional integro-differential equation

$$^cD^q u(t) = f\big(t, u(t), (\phi u)(t), (\psi u)(t)\big) \quad (0 < t < 1 \text{ and } 1 < q \leq 2)$$

with boundary values $u'(0) + au(\eta_1) = 0$, $bu'(1) + u(\eta_2) = 0$ and $0 < \eta_1 \leq \eta_2 < 1$, where $^cD^q$ is the Caputo fractional derivative, $a, b \in (0,1)$, $f : [0,1] \times X \times X \times X \to X$ is continuous and for the mappings $\gamma, \lambda : [0,1] \times [0,1] \to [0,\infty)$ with the property $\sup_{t\in[0,1]} |\int_0^t \lambda(t,s)\,ds| < \infty$ and $\sup_{t\in[0,1]} |\int_0^t \gamma(t,s)\,ds| < \infty$, the maps ϕ and ψ are defined by $(\phi u)(t) = \int_0^t \gamma(t,s)u(s)\,ds$ and $(\psi u)(t) = \int_0^t \lambda(t,s)u(s)\,ds$. Here, X is a Banach space (see [22]).

2 Main results

2.1 The basic problem

In this paper, we study the existence and uniqueness of solutions for the multi-term nonlinear fractional integro-differential equation

$$^cD^\alpha u(t) = f\big(t, u(t), (\phi u)(t), (\psi u)(t), {}^cD^{\beta_1}u(t), {}^cD^{\beta_2}u(t), \ldots, {}^cD^{\beta_n}u(t)\big) \quad (0 < t < 1) \tag{1}$$

with boundary values $u(0) + au(1) = 0$ and $u'(0) + bu'(1) = 0$, where $1 < \alpha < 2$, $0 < \beta_i < 1$, $\alpha - \beta_i \geq 1$, $a, b \neq -1$, $f : [0,1] \times \mathbb{R}^{n+3} \to \mathbb{R}$ is continuous, and for the mappings

$$\gamma, \lambda : [0,1] \times [0,1] \to [0,\infty)$$

with the property $\sup_{t\in[0,1]} |\int_0^t \lambda(t,s)\,ds| < \infty$ and $\sup_{t\in[0,1]} |\int_0^t \gamma(t,s)\,ds| < \infty$, the maps ϕ and ψ are defined by $(\phi u)(t) = \int_0^t \gamma(t,s)u(s)\,ds$ and $(\psi u)(t) = \int_0^t \lambda(t,s)u(s)\,ds$. In this way, we need the following result, which has been proved in [2].

Lemma 2.1 *Let $\alpha > 0$ and $n = [\alpha] + 1$. Then*

$$I^{\alpha}{}^cD^\alpha u(t) = u(t) + c_0 + c_1 t + c_2 t + \cdots + c_{n-1}t^{n-1},$$

where $c_0, c_1, \ldots, c_{n-1}$ are some real numbers.

The proof of the following result by using Lemma 2.1 is straightforward.

Lemma 2.2 *Let $y \in C[0,1]$, $a, b \neq -1$ and $1 < \alpha < 2$. Then the problem $^cD^\alpha u(t) = y(t)$ with boundary values $u(0) + au(1) = 0$ and $u'(0) + bu'(1) = 0$ has the unique solution*

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds - \frac{a}{(1+a)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\,ds \\ & + \frac{ab - b(1+a)t}{(1+a)(1+b)\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} y(s)\,ds. \end{aligned}$$

2.2 Some results on solving the problem

Let $C(I)$ be the space of all continuous real-valued functions on $I = [0,1]$ and

$$X = \big\{u : u \in C(I) \text{ and } {}^cD^{\beta_i}u \in C(I)\ (0 < \beta_i < 1) \text{ for } i = 1, 2, \ldots, n\big\}$$

endowed with the norm $\|u\| = \max_{t\in I} |u(t)| + \sum_{i=1}^n \max_{t\in I} |{}^cD^{\beta_i}u(t)|$. It is known that $(X, \|\cdot\|)$ is a Banach space.

Theorem 2.3 *Assume that there exist* $\kappa \in (0, \alpha - 1)$ *and* $\mu(t) \in L^{\frac{1}{\kappa}}([0,1],(0,\infty))$ *such that*

$$\begin{aligned}&\bigl|f(t,x,y,w,u_1,u_2,\dots,u_n) - f\bigl(t,x',y',w',v_1,v_2,\dots,v_n\bigr)\bigr|\\&\quad\le \mu(t)\bigl(\bigl|x-x'\bigr| + \bigl|y-y'\bigr| + \bigl|w-w'\bigr| + |u_1-v_1| + |u_2-v_2| + \cdots + |u_n-v_n|\bigr)\end{aligned}$$

for all $t \in [0,1]$ *and* $x,y,w,x',y',w',u_1,u_2,\dots,u_n,v_1,v_2,\dots,v_n \in \mathbb{R}$. *Then problem* (1) *has a unique solution whenever*

$$\begin{aligned}\Delta &= (1+\gamma_0+\lambda_0)\Biggl[\frac{(1+2|a|)\mu^*}{|1+a|\Gamma(\alpha)}\biggl(\frac{1-\kappa}{\alpha-\kappa}\biggr)^{1-\kappa} + \frac{|b|(1+2|a|)\mu^*}{|1+a||1+b|\Gamma(\alpha-1)}\biggl(\frac{1-\kappa}{\alpha-\kappa-1}\biggr)^{1-\kappa}\\&\quad+\sum_{i=1}^{n}\biggl(\frac{\Gamma(\alpha-\kappa)\mu^*}{\Gamma(\alpha-1)\Gamma(\alpha-\beta_i-\kappa+1)}\biggl(\frac{1-\kappa}{\alpha-\kappa-1}\biggr)^{1-\kappa}\\&\quad+\frac{|b|\mu^*}{|1+b|\Gamma(2-\beta_i)\Gamma(\alpha-1)}\biggl(\frac{1-\kappa}{\alpha-\kappa-1}\biggr)^{1-\kappa}\biggr)\Biggr] < 1,\end{aligned}$$

where $\gamma_0 = \sup_{t\in I}|\int_0^t \gamma(t,s)\,ds|$, $\lambda_0 = \sup_{t\in I}|\int_0^t \lambda(t,s)\,ds|$, $\mu^* = (\int_0^1 (\mu(s))^{\frac{1}{\kappa}}\,ds)^{\kappa}$.

Proof Define the mapping $F : X \to X$ by

$$\begin{aligned}(Fu)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,u(s),(\phi u)(s),(\psi u)(s),{}^cD^{\beta_1}u(s),{}^cD^{\beta_2}u(s),\dots,{}^cD^{\beta_n}u(s)\bigr)\,ds\\&\quad-\frac{a}{(1+a)}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\\&\quad\times f\bigl(s,u(s),(\phi u)(s),(\psi u)(s),{}^cD^{\beta_1}u(s),{}^cD^{\beta_2}u(s),\dots,{}^cD^{\beta_n}u(s)\bigr)\,ds\\&\quad+\frac{ab-b(1+a)t}{(1+a)(1+b)}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\\&\quad\times f\bigl(s,u(s),(\phi u)(s),(\psi u)(s),{}^cD^{\beta_1}u(s),{}^cD^{\beta_2}u(s),\dots,{}^cD^{\beta_n}u(s)\bigr)\,ds.\end{aligned}$$

For each $u, v \in X$ and $t \in [0,1]$, by using the Hölder inequality, we have

$$\begin{aligned}&\bigl|(Fu)(t)-(Fv)(t)\bigr|\\&\quad=\biggl|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl(f\bigl(s,u(s),(\phi u)(s),(\psi u)(s),{}^cD^{\beta_1}u(s),{}^cD^{\beta_2}u(s),\dots,{}^cD^{\beta_n}u(s)\bigr)\\&\qquad-f\bigl(s,v(s),(\phi v)(s),(\psi v)(s),{}^cD^{\beta_1}v(s),{}^cD^{\beta_2}v(s),\dots,{}^cD^{\beta_n}v(s)\bigr)\bigr)\,ds\\&\qquad-\frac{a}{(1+a)}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl(f\bigl(s,u(s),(\phi u)(s),(\psi u)(s),{}^cD^{\beta_1}u(s),{}^cD^{\beta_2}u(s),\dots,{}^cD^{\beta_n}u(s)\bigr)\\&\qquad-f\bigl(s,v(s),(\phi v)(s),(\psi v)(s),{}^cD^{\beta_1}v(s),{}^cD^{\beta_2}v(s),\dots,{}^cD^{\beta_n}v(s)\bigr)\bigr)\,ds\\&\qquad+\frac{ab-b(1+a)t}{(1+a)(1+b)}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\bigl(f\bigl(s,u(s),(\phi u)(s),(\psi u)(s),{}^cD^{\beta_1}u(s),{}^cD^{\beta_2}u(s),\\&\qquad\dots,{}^cD^{\beta_n}u(s)\bigr)-f\bigl(s,v(s),(\phi v)(s),(\psi v)(s),{}^cD^{\beta_1}v(s),{}^cD^{\beta_2}v(s),\dots,{}^cD^{\beta_n}v(s)\bigr)\bigr)\,ds\biggr|\\&\quad\le\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(s,u(s),(\phi u)(s),(\psi u)(s),{}^cD^{\beta_1}u(s),{}^cD^{\beta_2}u(s),\dots,{}^cD^{\beta_n}u(s)\bigr)\end{aligned}$$

$$
\begin{aligned}
&-f\bigl(s,v(s),(\phi v)(s),(\psi v)(s),{}^cD^{\beta_1}v(s),{}^cD^{\beta_2}v(s),\ldots,{}^cD^{\beta_n}v(s)\bigr)\bigr|\,ds\\
&+\frac{|a|}{|1+a|}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(s,u(s),(\phi u)(s),(\psi u)(s),{}^cD^{\beta_1}u(s),{}^cD^{\beta_2}u(s),\ldots,{}^cD^{\beta_n}u(s)\bigr)\\
&-f\bigl(s,v(s),(\phi v)(s),(\psi v)(s),{}^cD^{\beta_1}v(s),{}^cD^{\beta_2}v(s),\ldots,{}^cD^{\beta_n}v(s)\bigr)\bigr|\,ds\\
&+\frac{|ab-b(1+a)t|}{|1+a||1+b|}\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\bigl|f\bigl(s,u(s),(\phi u)(s),(\psi u)(s),{}^cD^{\beta_1}u(s),{}^cD^{\beta_2}u(s),\\
&\ldots,{}^cD^{\beta_n}u(s)\bigr)-f\bigl(s,v(s),(\phi v)(s),(\psi v)(s),{}^cD^{\beta_1}v(s),{}^cD^{\beta_2}v(s),\ldots,{}^cD^{\beta_n}v(s)\bigr)\bigr|\,ds\\
\le{}&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\mu(s)\bigl(\bigl|u(s)-v(s)\bigr|+\bigl|(\phi u)(s)-(\phi v)(s)\bigr|+\bigl|(\psi u)(s)-(\psi v)(s)\bigr|\\
&+\bigl|{}^cD^{\beta_1}u(s)-{}^cD^{\beta_1}v(s)\bigr|+\bigl|{}^cD^{\beta_2}u(s)-{}^cD^{\beta_2}v(s)\bigr|+\cdots+\bigl|{}^cD^{\beta_n}u(s)-{}^cD^{\beta_n}v(s)\bigr|\bigr)\,ds\\
&+\frac{|a|}{|1+a|}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\mu(s)\bigl(\bigl|u(s)-v(s)\bigr|+\bigl|(\phi u)(s)-(\phi v)(s)\bigr|+\bigl|(\psi u)(s)-(\psi v)(s)\bigr|\\
&+\bigl|{}^cD^{\beta_1}u(s)-{}^cD^{\beta_1}v(s)\bigr|+\bigl|{}^cD^{\beta_2}u(s)-{}^cD^{\beta_2}v(s)\bigr|+\cdots+\bigl|{}^cD^{\beta_n}u(s)-{}^cD^{\beta_n}v(s)\bigr|\bigr)\,ds\\
&+\frac{|b|(1+2|a|)}{|1+a||1+b|}\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\mu(s)\bigl(\bigl|u(s)-v(s)\bigr|+\bigl|(\phi u)(s)-(\phi v)(s)\bigr|\\
&+\bigl|(\psi u)(s)-(\psi v)(s)\bigr|+\bigl|{}^cD^{\beta_1}u(s)-{}^cD^{\beta_1}v(s)\bigr|\\
&+\bigl|{}^cD^{\beta_2}u(s)-{}^cD^{\beta_2}v(s)\bigr|+\cdots+\bigl|{}^cD^{\beta_n}u(s)-{}^cD^{\beta_n}v(s)\bigr|\bigr)\,ds\\
\le{}&\frac{(1+\gamma_0+\lambda_0)\|u-v\|}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\mu(s)\,ds\\
&+\frac{|a|(1+\gamma_0+\lambda_0)\|u-v\|}{|1+a|\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\mu(s)\,ds\\
&+\frac{|b|(1+2|a|)(1+\gamma_0+\lambda_0)\|u-v\|}{|1+a||1+b|\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}\mu(s)\,ds\\
\le{}&\frac{(1+\gamma_0+\lambda_0)\|u-v\|}{\Gamma(\alpha)}\biggl(\int_0^t\bigl((t-s)^{\alpha-1}\bigr)^{\frac{1}{1-\kappa}}\,ds\biggr)^{1-\kappa}\biggl(\int_0^t\bigl(\mu(s)\bigr)^{\frac{1}{\kappa}}\,ds\biggr)^{\kappa}\\
&+\frac{|a|(1+\gamma_0+\lambda_0)\|u-v\|}{|1+a|\Gamma(\alpha)}\biggl(\int_0^1\bigl((1-s)^{\alpha-1}\bigr)^{\frac{1}{1-\kappa}}\,ds\biggr)^{1-\kappa}\biggl(\int_0^1\bigl(\mu(s)\bigr)^{\frac{1}{\kappa}}\,ds\biggr)^{\kappa}\\
&+\frac{|b|(1+2|a|)(1+\gamma_0+\lambda_0)\|u-v\|}{|1+a||1+b|\Gamma(\alpha-1)}\\
&\times\biggl(\int_0^1\bigl((1-s)^{\alpha-2}\bigr)^{\frac{1}{1-\kappa}}\,ds\biggr)^{1-\kappa}\biggl(\int_0^1\bigl(\mu(s)\bigr)^{\frac{1}{\kappa}}\,ds\biggr)^{\kappa}\\
\le{}&\frac{\mu^*(1+\gamma_0+\lambda_0)\|u-v\|}{\Gamma(\alpha)}\biggl(\frac{1-\kappa}{\alpha-\kappa}\biggr)^{1-\kappa}+\frac{|a|\mu^*(1+\gamma_0+\lambda_0)\|u-v\|}{|1+a|\Gamma(\alpha)}\biggl(\frac{1-\kappa}{\alpha-\kappa}\biggr)^{1-\kappa}\\
&+\frac{|b|(1+2|a|)\mu^*(1+\gamma_0+\lambda_0)\|u-v\|}{|1+a||1+b|\Gamma(\alpha-1)}\biggl(\frac{1-\kappa}{\alpha-\kappa-1}\biggr)^{1-\kappa}\\
\le{}&(1+\gamma_0+\lambda_0)\biggl[\frac{(1+2|a|)\mu^*}{|1+a|\Gamma(\alpha)}\biggl(\frac{1-\kappa}{\alpha-\kappa}\biggr)^{1-\kappa}\\
&+\frac{|b|(1+2|a|)\mu^*}{|1+a||1+b|\Gamma(\alpha-1)}\biggl(\frac{1-\kappa}{\alpha-\kappa-1}\biggr)^{1-\kappa}\biggr]\|u-v\|.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
&\left|{}^cD^{\beta_i}(Fu)(t) - {}^cD^{\beta_i}(Fv)(t)\right| \\
&= \left|\int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)}(Fu)'(s)\,ds - \int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)}(Fv)'(s)\,ds\right| \\
&= \left|\int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\right.\right. \\
&\quad \times f\bigl(\tau, u(\tau), (\phi u)(\tau), (\psi u)(\tau), {}^cD^{\beta_1}u(\tau), {}^cD^{\beta_2}u(\tau), \ldots, {}^cD^{\beta_n}u(\tau)\bigr)\,d\tau \\
&\quad - \frac{b}{1+b}\int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \\
&\quad \left.\times f\bigl(\tau, u(\tau), (\phi u)(\tau), (\psi u)(\tau), {}^cD^{\beta_1}u(\tau), {}^cD^{\beta_2}u(\tau), \ldots, {}^cD^{\beta_n}u(\tau)\bigr)\,d\tau\right)ds \\
&\quad - \int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\right. \\
&\quad \times f\bigl(\tau, v(\tau), (\phi v)(\tau), (\psi v)(\tau), {}^cD^{\beta_1}v(\tau), {}^cD^{\beta_2}v(\tau), \ldots, {}^cD^{\beta_n}v(\tau)\bigr)\,d\tau \\
&\quad - \frac{b}{1+b}\int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \\
&\quad \left.\left.\times f\bigl(\tau, v(\tau), (\phi v)(\tau), (\psi v)(\tau), {}^cD^{\beta_1}v(\tau), {}^cD^{\beta_2}v(\tau), \ldots, {}^cD^{\beta_n}v(\tau)\bigr)\,d\tau\right)ds\right| \\
&\le \int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\right. \\
&\quad \times \bigl|f\bigl(\tau, u(\tau), (\phi u)(\tau), (\psi u)(\tau), {}^cD^{\beta_1}u(\tau), {}^cD^{\beta_2}u(\tau), \ldots, {}^cD^{\beta_n}u(\tau)\bigr) \\
&\quad \left. - f\bigl(\tau, v(\tau), (\phi v)(\tau), (\psi v)(\tau), {}^cD^{\beta_1}v(\tau), {}^cD^{\beta_2}v(\tau), \ldots, {}^cD^{\beta_n}v(\tau)\bigr)\bigr|\,d\tau\right)ds \\
&\quad + \frac{|b|}{|1+b|}\int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)} \\
&\quad \times \left(\int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\bigl|f\bigl(\tau, u(\tau), (\phi u)(\tau), (\psi u)(\tau), {}^cD^{\beta_1}u(\tau), {}^cD^{\beta_2}u(\tau), \ldots, {}^cD^{\beta_n}u(\tau)\bigr)\right. \\
&\quad \left. - f\bigl(\tau, v(\tau), (\phi v)(\tau), (\psi v)(\tau), {}^cD^{\beta_1}v(\tau), {}^cD^{\beta_2}v(\tau), \ldots, {}^cD^{\beta_n}v(\tau)\bigr)\bigr|\,d\tau\right)ds \\
&\le \frac{(1+\gamma_0+\lambda_0)\|u-v\|}{\Gamma(1-\beta_i)\Gamma(\alpha-1)}\int_0^t (t-s)^{-\beta_i}\left(\int_0^s (s-\tau)^{\alpha-2}\mu(\tau)\,d\tau\right)ds \\
&\quad + \frac{|b|(1+\gamma_0+\lambda_0)\|u-v\|}{|1+b|\Gamma(1-\beta_i)\Gamma(\alpha-1)}\int_0^t (t-s)^{-\beta_i}\left(\int_0^1 (1-\tau)^{\alpha-2}\mu(\tau)\,d\tau\right)ds \\
&\le \frac{\mu^*(1+\gamma_0+\lambda_0)\|u-v\|}{\Gamma(1-\beta_i)\Gamma(\alpha-1)}\left(\frac{1-\kappa}{\alpha-\kappa-1}\right)^{1-\kappa}\int_0^t (t-s)^{-\beta_i}s^{\alpha-\kappa-1}\,ds \\
&\quad + \frac{|b|\mu^*(1+\gamma_0+\lambda_0)\|u-v\|}{|1+b|\Gamma(1-\beta_i)\Gamma(\alpha-1)}\left(\frac{1-\kappa}{\alpha-\kappa-1}\right)^{1-\kappa}\int_0^t (t-s)^{-\beta_i}\,ds \\
&\le \frac{\mu^*(1+\gamma_0+\lambda_0)\|u-v\|}{\Gamma(1-\beta_i)\Gamma(\alpha-1)}\left(\frac{1-\kappa}{\alpha-\kappa-1}\right)^{1-\kappa}\int_0^1 (1-\xi)^{-\beta_i}\xi^{\alpha-\kappa-1}\,d\xi \\
&\quad + \frac{|b|\mu^*(1+\gamma_0+\lambda_0)\|u-v\|}{|1+b|\Gamma(2-\beta_i)\Gamma(\alpha-1)}\left(\frac{1-\kappa}{\alpha-\kappa-1}\right)^{1-\kappa}.
\end{aligned}
$$

Since $B(\alpha-\kappa,1-\beta_i)=\int_0^1(1-\xi)^{-\beta_i}\xi^{\alpha-\kappa-1}\,d\xi=\frac{\Gamma(\alpha-\kappa)\Gamma(1-\beta_i)}{\Gamma(\alpha-\beta_i-\kappa+1)}$, we obtain

$$\begin{aligned}\left|{}^cD^{\beta_i}(Fu)(t)-{}^cD^{\beta_i}(Fv)(t)\right| &\le (1+\gamma_0+\lambda_0)\left[\frac{\Gamma(\alpha-\kappa)\mu^*}{\Gamma(\alpha-1)\Gamma(\alpha-\beta_i-\kappa+1)}\left(\frac{1-\kappa}{\alpha-\kappa-1}\right)^{1-\kappa}\right.\\ &\quad\left.+\frac{|b|\mu^*}{|1+b|\Gamma(2-\beta_i)\Gamma(\alpha-1)}\left(\frac{1-\kappa}{\alpha-\kappa-1}\right)^{1-\kappa}\right]\|u-v\|\end{aligned}$$

for all $i=1,2,\ldots,n$. Hence, we get

$$\begin{aligned}&\|Fu-Fv\|\\ &\quad\le(1+\gamma_0+\lambda_0)\left[\frac{(1+2|a|)\mu^*}{|1+a|\Gamma(\alpha)}\left(\frac{1-\kappa}{\alpha-\kappa}\right)^{1-\kappa}+\frac{|b|(1+2|a|)\mu^*}{|1+a||1+b|\Gamma(\alpha-1)}\left(\frac{1-\kappa}{\alpha-\kappa-1}\right)^{1-\kappa}\right.\\ &\qquad+\sum_{i=1}^{n}\left(\frac{\Gamma(\alpha-\kappa)\mu^*}{\Gamma(\alpha-1)\Gamma(\alpha-\beta_i-\kappa+1)}\left(\frac{1-\kappa}{\alpha-\kappa-1}\right)^{1-\kappa}\right.\\ &\qquad\left.\left.+\frac{|b|\mu^*}{|1+b|\Gamma(2-\beta_i)\Gamma(\alpha-1)}\left(\frac{1-\kappa}{\alpha-\kappa-1}\right)^{1-\kappa}\right)\right]\|u-v\|=\Delta\|u-v\|.\end{aligned}$$

Since $\Delta<1$, F is a contraction mapping, therefore, by using the Banach contraction principle, F has a unique fixed point, which is the unique solution of problem (1) by using Lemma 2.2. □

Corollary 2.4 *Assume that there exists $L>0$ such that*

$$\begin{aligned}&\left|f(t,x,y,w,u_1,u_2,\ldots,u_n)-f\left(t,x',y',w',v_1,v_2,\ldots,v_n\right)\right|\\ &\quad\le L\left(\left|x-x'\right|+\left|y-y'\right|+\left|w-w'\right|+|u_1-v_1|+|u_2-v_2|+\cdots+|u_n-v_n|\right)\end{aligned}$$

for all $t\in[0,1]$ and $x,y,w,x',y',w',u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n\in\mathbb{R}$. Then problem (1) *has a unique solution whenever*

$$\begin{aligned}&(1+\gamma_0+\lambda_0)\left[\frac{(1+2|a|)(1+(\alpha+1)|b|)L}{|1+a||1+b|\Gamma(\alpha+1)}\right.\\ &\quad\left.+\sum_{i=1}^{n}\left(\frac{L}{\Gamma(\alpha-\beta_i+1)}+\frac{|b|L}{|1+b|\Gamma(2-\beta_i)\Gamma(\alpha)}\right)\right]<1,\end{aligned}$$

where $\gamma_0=\sup_{t\in I}|\int_0^t\gamma(t,s)\,ds|$, $\lambda_0=\sup_{t\in I}|\int_0^t\lambda(t,s)\,ds|$.

Now, we restate the Schauder's fixed point theorem, which is needed to prove next result (see Theorem 1.10.16 in [23]).

Theorem 2.5 *Let E be a closed, convex and bounded subset of a Banach space X, and let $F:E\to E$ be a continuous mapping such that $F(E)$ is a relatively compact subset of X. Then F has a fixed point in E.*

Theorem 2.6 *Let $f : [0,1] \times \mathbb{R}^{n+3} \to \mathbb{R}$ be a continuous function such that there exists a constant $l \in (0, \alpha - 1)$ and a real-valued function $m(t) \in L^{\frac{1}{l}}([0,1], (0,\infty))$ such that*

$$\begin{aligned}
&\bigl|f(t,x,y,w,u_1,u_2,\ldots,u_n)\bigr| \\
&\quad \leq m(t) + d|x|^{\rho} + d'|y|^{\rho'} + d''|w|^{\rho''} + d_1|u_1|^{\rho_1} + d_2|u_2|^{\rho_2} + \cdots + d_n|u_n|^{\rho_n}, \qquad (*)
\end{aligned}$$

where $d, d', d'', d_i \geq 0$ and $0 < \rho, \rho', \rho'', \rho_i < 1$ for $i = 1, 2, \ldots, n$, or

$$\begin{aligned}
&\bigl|f(t,x,y,w,u_1,u_2,\ldots,u_n)\bigr| \\
&\quad \leq d|x|^{\rho} + d'|y|^{\rho'} + d''|w|^{\rho''} + d_1|u_1|^{\rho_1} + d_2|u_2|^{\rho_2} + \cdots + d_n|u_n|^{\rho_n},
\end{aligned}$$

where $d, d', d'', d_i > 0$ and $\rho, \rho', \rho'', \rho_i > 1$ for $i = 1, 2, \ldots, n$. Then problem (1) *has a solution.*

Proof First, suppose that f satisfy condition $(*)$. Define $B_r = \{u \in X, \|u\| \leq r\}$, where

$$\begin{aligned}
r \geq{}& \max\Bigl\{\bigl((n+4)Ad\bigr)^{\frac{1}{1-\rho}}, \bigl((n+4)Ad'\gamma_0^{p'}\bigr)^{\frac{1}{1-\rho'}}, \bigl((n+4)Ad''\lambda_0^{p''}\bigr)^{\frac{1}{1-\rho''}}, \bigl((n+4)Ad_1\bigr)^{\frac{1}{1-\rho_1}}, \\
&\bigl((n+4)Ad_2\bigr)^{\frac{1}{1-\rho_2}}, \ldots, \bigl((n+4)Ad_n\bigr)^{\frac{1}{1-\rho_n}}, (n+4)K\Bigr\}, \\
K ={}& \frac{(1+2|a|)M}{|1+a|\Gamma(\alpha)}\left(\frac{1-l}{\alpha-l}\right)^{1-l} + \frac{|b|(1+2|a|)M}{|1+a||1+b|\Gamma(\alpha-1)}\left(\frac{1-l}{\alpha-l-1}\right)^{1-l} \\
&+ \sum_{i=1}^{n}\left(\frac{\Gamma(\alpha-l)M}{\Gamma(\alpha-1)\Gamma(\alpha-\beta_i-l+1)}\left(\frac{1-l}{\alpha-l-1}\right)^{1-l}\right. \\
&\left.+ \frac{|b|M}{|1+b|\Gamma(2-\beta_i)\Gamma(\alpha-1)}\left(\frac{1-l}{\alpha-l-1}\right)^{1-l}\right), \\
A ={}& \frac{(1+2|a|)(1+(1+\alpha)|b|)}{|1+a||1+b|\Gamma(\alpha+1)} + \sum_{i=1}^{n}\left(\frac{1}{\Gamma(\alpha-\beta_i+1)} + \frac{|b|}{|1+b|\Gamma(\alpha)\Gamma(2-\beta_i)}\right)
\end{aligned}$$

and $M = (\int_0^1 (m(t))^{\frac{1}{l}}\,ds)^l$. Note that B_r is a closed, bounded and convex subset of the Banach space X. For each $u \in B_r$, we have

$$\begin{aligned}
\bigl|(Fu)(t)\bigr| ={}& \biggl|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,u(s),(\phi u)(s),(\psi u)(s),{}^cD^{\beta_1}u(s),{}^cD^{\beta_2}u(s),\ldots,{}^cD^{\beta_n}u(s)\bigr)\,ds \\
&- \frac{a}{(1+a)}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\
&\times f\bigl(s,u(s),(\phi u)(s),(\psi u)(s),{}^cD^{\beta_1}u(s),{}^cD^{\beta_2}u(s),\ldots,{}^cD^{\beta_n}u(s)\bigr)\,ds \\
&+ \frac{ab-b(1+a)t}{(1+a)(1+b)}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\
&\times f\bigl(s,u(s),(\phi u)(s),(\psi u)(s),{}^cD^{\beta_1}u(s),{}^cD^{\beta_2}u(s),\ldots,{}^cD^{\beta_n}u(s)\bigr)\,ds\biggr| \\
\leq{}& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(s,u(s),(\phi u)(s),(\psi u)(s),{}^cD^{\beta_1}u(s),{}^cD^{\beta_2}u(s),\ldots,{}^cD^{\beta_n}u(s)\bigr)\bigr|\,ds \\
&+ \frac{|a|}{|1+a|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}
\end{aligned}$$

$$
\begin{aligned}
&\times \bigl|f\bigl(s,u(s),(\phi u)(s),(\psi u)(s),{}^cD^{\beta_1}u(s),{}^cD^{\beta_2}u(s),\ldots,{}^cD^{\beta_n}u(s)\bigr)\bigr|\,ds \\
&+ \frac{|b|(1+2|a|)}{|1+a||1+b|}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\
&\times \bigl|f\bigl(s,u(s),(\phi u)(s),(\psi u)(s),{}^cD^{\beta_1}u(s),{}^cD^{\beta_2}u(s),\ldots,{}^cD^{\beta_n}u(s)\bigr)\bigr|\,ds \\
\le{}& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)\,ds \\
&+ \bigl(dr^{\rho} + d'\gamma_0^{p'}r^{\rho'} + d''\lambda_0^{p''}r^{\rho''} + d_1r^{\rho_1} + d_2r^{\rho_2} + \cdots + d_nr^{\rho_n}\bigr)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds \\
&+ \frac{|a|}{|1+a|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)\,ds \\
&+ \frac{|a|}{|1+a|}\bigl(dr^{\rho} + d'\gamma_0^{p'}r^{\rho'} + d''\lambda_0^{p''}r^{\rho''} + d_1r^{\rho_1} + d_2r^{\rho_2} + \cdots + d_nr^{\rho_n}\bigr) \\
&\times \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds + \frac{|b|(1+2|a|)}{|1+a||1+b|}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} m(s)\,ds \\
&+ \frac{|b|(1+2|a|)}{|1+a||1+b|}\bigl(dr^{\rho} + d'\gamma_0^{p'}r^{\rho'} + d''\lambda_0^{p''}r^{\rho''} + d_1r^{\rho_1} + d_2r^{\rho_2} + \cdots + d_nr^{\rho_n}\bigr) \\
&\times \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\,ds \\
\le{}& \frac{1}{\Gamma(\alpha)}\biggl(\int_0^t \bigl((t-s)^{\alpha-1}\bigr)^{\frac{1}{1-l}}\,ds\biggr)^{1-l}\biggl(\int_0^t \bigl(m(s)\bigr)^{\frac{1}{l}}\,ds\biggr)^{l} \\
&+ \frac{|a|}{|1+a|\Gamma(\alpha)}\biggl(\int_0^1 \bigl((1-s)^{\alpha-1}\bigr)^{\frac{1}{1-l}}\,ds\biggr)^{1-l}\biggl(\int_0^1 \bigl(m(s)\bigr)^{\frac{1}{l}}\,ds\biggr)^{l} \\
&+ \frac{|b|(1+2|a|)}{|1+a||1+b|\Gamma(\alpha-1)}\biggl(\int_0^1 \bigl((1-s)^{\alpha-2}\bigr)^{\frac{1}{1-l}}\,ds\biggr)^{1-l}\biggl(\int_0^1 \bigl(m(s)\bigr)^{\frac{1}{l}}\,ds\biggr)^{l} \\
&+ \frac{(1+2|a|)(1+(1+\alpha)|b|)}{|1+a||1+b|\Gamma(\alpha+1)} \\
&\times \bigl(dr^{\rho} + d'\gamma_0^{p'}r^{\rho'} + d''\lambda_0^{p''}r^{\rho''} + d_1r^{\rho_1} + d_2r^{\rho_2} + \cdots + d_nr^{\rho_n}\bigr) \\
\le{}& \frac{(1+2|a|)M}{|1+a|\Gamma(\alpha)}\biggl(\frac{1-l}{\alpha-l}\biggr)^{1-l} + \frac{|b|(1+2|a|)M}{|1+a||1+b|\Gamma(\alpha-1)}\biggl(\frac{1-l}{\alpha-l-1}\biggr)^{1-l} \\
&+ \frac{(1+2|a|)(1+(1+\alpha)|b|)}{|1+a||1+b|\Gamma(\alpha+1)} \\
&\times \bigl(dr^{\rho} + d'\gamma_0^{p'}r^{\rho'} + d''\lambda_0^{p''}r^{\rho''} + d_1r^{\rho_1} + d_2r^{\rho_2} + \cdots + d_nr^{\rho_n}\bigr).
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
\bigl|{}^cD^{\beta_i}(Fu)(t)\bigr| &= \biggl|\int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)}(Fu)'(s)\,ds\biggr| \\
&= \biggl|\int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)}\biggl(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \\
&\quad \times f\bigl(\tau,u(\tau),(\phi u)(\tau),(\psi u)(\tau),{}^cD^{\beta_1}u(\tau),{}^cD^{\beta_2}u(\tau),\ldots,{}^cD^{\beta_n}u(\tau)\bigr)\,d\tau \\
&\quad - \frac{b}{1+b}\int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}
\end{aligned}
$$

$$
\begin{aligned}
&\times f\bigl(\tau, u(\tau), (\phi u)(\tau), (\psi u)(\tau), \\
&{}^cD^{\beta_1}u(\tau), {}^cD^{\beta_2}u(\tau), \dots, {}^cD^{\beta_n}u(\tau)\bigr)\,d\tau\biggr)\,ds\biggr| \\
\le{}& \int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)}\biggl(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\bigl|f\bigl(\tau, u(\tau), (\phi u)(\tau), (\psi u)(\tau), \\
&{}^cD^{\beta_1}u(\tau), {}^cD^{\beta_2}u(\tau), \dots, {}^cD^{\beta_n}u(\tau)\bigr)\bigr|\,d\tau\biggr)\,ds \\
&+ \frac{|b|}{|1+b|}\int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)}\biggl(\int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\bigl|f\bigl(\tau, u(\tau), (\phi u)(\tau), (\psi u)(\tau), \\
&{}^cD^{\beta_1}u(\tau), {}^cD^{\beta_2}u(\tau), \dots, {}^cD^{\beta_n}u(\tau)\bigr)\bigr|\,d\tau\biggr)\,ds \\
\le{}& \int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)}\biggl(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}m(\tau)\,d\tau\biggr)\,ds \\
&+ \bigl(dr^{\rho} + d'\gamma_0^{p'}r^{\rho'} + d''\lambda_0^{p''}r^{\rho''} + d_1r^{\rho_1} + d_2r^{\rho_2} + \cdots + d_nr^{\rho_n}\bigr) \\
&\times \int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)}\biggl(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\,d\tau\biggr)\,ds \\
&+ \frac{|b|}{|1+b|}\int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)}\biggl(\int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}m(\tau)\,d\tau\biggr)\,ds \\
&+ \frac{|b|}{|1+b|}\bigl(dr^{\rho} + d'\gamma_0^{p'}r^{\rho'} + d''\lambda_0^{p''}r^{\rho''} + d_1r^{\rho_1} + d_2r^{\rho_2} + \cdots + d_nr^{\rho_n}\bigr) \\
&\times \int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)}\biggl(\int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\,d\tau\biggr)\,ds \\
\le{}& \frac{1}{\Gamma(\alpha-1)\Gamma(1-\beta_i)}\int_0^t (t-s)^{-\beta_i} \\
&\times \biggl[\biggl(\int_0^s \bigl((s-\tau)^{\alpha-2}\bigr)^{\frac{1}{1-l}}\,d\tau\biggr)^{1-l}\biggl(\int_0^s \bigl(m(\tau)\bigr)^{\frac{1}{l}}\,d\tau\biggr)^{l}\biggr]\,ds \\
&+ \frac{(dr^{\rho} + d'\gamma_0^{p'}r^{\rho'} + d''\lambda_0^{p''}r^{\rho''} + d_1r^{\rho_1} + d_2r^{\rho_2} + \cdots + d_nr^{\rho_n})}{\Gamma(\alpha)\Gamma(1-\beta_i)} \\
&\times \int_0^t (t-s)^{-\beta_i}s^{\alpha-1}\,ds + \frac{|b|}{|1+b|\Gamma(\alpha-1)\Gamma(1-\beta_i)} \\
&\times \int_0^t (t-s)^{-\beta_i}\biggl[\biggl(\int_0^1 \bigl((1-\tau)^{\alpha-2}\bigr)^{\frac{1}{1-l}}\,d\tau\biggr)^{1-l}\biggl(\int_0^1 \bigl(m(\tau)\bigr)^{\frac{1}{l}}\,d\tau\biggr)^{l}\biggr]\,ds \\
&+ \frac{|b|(dr^{\rho} + d'\gamma_0^{p'}r^{\rho'} + d''\lambda_0^{p''}r^{\rho''} + d_1r^{\rho_1} + d_2r^{\rho_2} + \cdots + d_nr^{\rho_n})}{|1+b|\Gamma(\alpha)\Gamma(2-\beta_i)} \\
\le{}& \frac{M}{\Gamma(\alpha-1)\Gamma(1-\beta_i)}\biggl(\frac{1-l}{\alpha-l-1}\biggr)^{1-l}\int_0^t (t-s)^{-\beta_i}s^{\alpha-l-1}\,ds \\
&+ \frac{(dr^{\rho} + d'\gamma_0^{p'}r^{\rho'} + d''\lambda_0^{p''}r^{\rho''} + d_1r^{\rho_1} + d_2r^{\rho_2} + \cdots + d_nr^{\rho_n})}{\Gamma(\alpha)\Gamma(1-\beta_i)} \\
&\times \int_0^t (t-s)^{-\beta_i}s^{\alpha-1}\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{|b|M}{|1+b|\Gamma(\alpha-1)\Gamma(1-\beta_i)}\left(\frac{1-l}{\alpha-l-1}\right)^{1-l}\int_0^t (t-s)^{-\beta_i}\,ds \\
&+ \frac{|b|(dr^\rho + d'\gamma_0^{p'}r^{\rho'} + d''\lambda_0^{p''}r^{\rho''} + d_1r^{\rho_1} + d_2r^{\rho_2} + \cdots + d_nr^{\rho_n})}{|1+b|\Gamma(\alpha)\Gamma(2-\beta_i)} \\
\le{}& \frac{M}{\Gamma(\alpha-1)\Gamma(1-\beta_i)}\left(\frac{1-l}{\alpha-l-1}\right)^{1-l}\int_0^1 (1-\xi)^{-\beta_i}\xi^{\alpha-l-1}\,d\xi \\
&+ \frac{(dr^\rho + d'\gamma_0^{p'}r^{\rho'} + d''\lambda_0^{p''}r^{\rho''} + d_1r^{\rho_1} + d_2r^{\rho_2} + \cdots + d_nr^{\rho_n})}{\Gamma(\alpha)\Gamma(1-\beta_i)} \\
&\times \int_0^1 (1-\xi)^{-\beta_i}\xi^{\alpha-1}\,d\xi \\
&+ \frac{|b|M}{|1+b|\Gamma(\alpha-1)\Gamma(2-\beta_i)}\left(\frac{1-l}{\alpha-l-1}\right)^{1-l} \\
&+ \frac{|b|(dr^\rho + d'\gamma_0^{p'}r^{\rho'} + d''\lambda_0^{p''}r^{\rho''} + d_1r^{\rho_1} + d_2r^{\rho_2} + \cdots + d_nr^{\rho_n})}{|1+b|\Gamma(\alpha)\Gamma(2-\beta_i)}.
\end{aligned}
$$

Since $B(\alpha-l,1-\beta_i) = \int_0^1 (1-\xi)^{-\beta_i}\xi^{\alpha-l-1}\,d\xi = \frac{\Gamma(\alpha-l)\Gamma(1-\beta_i)}{\Gamma(\alpha-\beta_i-l+1)}$ and, on the other hand, $B(\alpha,1-\beta_i) = \int_0^1 (1-\xi)^{-\beta_i}\xi^{\alpha-1}\,d\xi = \frac{\Gamma(\alpha)\Gamma(1-\beta_i)}{\Gamma(\alpha-\beta_i+1)}$, we conclude that

$$
\begin{aligned}
\left|{}^cD^{\beta_i}(Fu)(t)\right| \le{}& \frac{\Gamma(\alpha-l)M}{\Gamma(\alpha-1)\Gamma(\alpha-\beta_i-l+1)}\left(\frac{1-l}{\alpha-l-1}\right)^{1-l} \\
&+ \frac{|b|M}{|1+b|\Gamma(\alpha-1)\Gamma(2-\beta_i)}\left(\frac{1-l}{\alpha-l-1}\right)^{1-l} \\
&+ \frac{dr^\rho + d'\gamma_0^{p'}r^{\rho'} + d''\lambda_0^{p''}r^{\rho''} + d_1r^{\rho_1} + d_2r^{\rho_2} + \cdots + d_nr^{\rho_n}}{\Gamma(\alpha-\beta_i+1)} \\
&+ \frac{|b|(dr^\rho + d'\gamma_0^{p'}r^{\rho'} + d''\lambda_0^{p''}r^{\rho''} + d_1r^{\rho_1} + d_2r^{\rho_2} + \cdots + d_nr^{\rho_n})}{|1+b|\Gamma(\alpha)\Gamma(2-\beta_i)}
\end{aligned}
$$

for all $i = 1,2,\ldots,n$. Thus,

$$
\begin{aligned}
\|Fu\| \le{}& \frac{(1+2|a|)M}{|1+a|\Gamma(\alpha)}\left(\frac{1-l}{\alpha-l}\right)^{1-l} + \frac{|b|(1+2|a|)M}{|1+a||1+b|\Gamma(\alpha-1)}\left(\frac{1-l}{\alpha-l-1}\right)^{1-l} \\
&+ \sum_{i=1}^n \left[\frac{\Gamma(\alpha-l)M}{\Gamma(\alpha-1)\Gamma(\alpha-\beta_i-l+1)}\left(\frac{1-l}{\alpha-l-1}\right)^{1-l}\right. \\
&\left.+ \frac{|b|M}{|1+b|\Gamma(\alpha-1)\Gamma(2-\beta_i)}\left(\frac{1-l}{\alpha-l-1}\right)^{1-l}\right] \\
&+ \left(dr^\rho + d'\gamma_0^{p'}r^{\rho'} + d''\lambda_0^{p''}r^{\rho''} + d_1r^{\rho_1} + d_2r^{\rho_2} + \cdots + d_nr^{\rho_n}\right) \\
&\times \left(\frac{(1+2|a|)(1+(1+\alpha)|b|)}{|1+a||1+b|\Gamma(\alpha+1)} + \sum_{i=1}^n \left[\frac{1}{\Gamma(\alpha-\beta_i+1)} + \frac{|b|}{|1+b|\Gamma(\alpha)\Gamma(2-\beta_i)}\right]\right) \\
={}& K + \left(dr^\rho + d'\gamma_0^{p'}r^{\rho'} + d''\lambda_0^{p''}r^{\rho''} + d_1r^{\rho_1} + d_2r^{\rho_2} + \cdots + d_nr^{\rho_n}\right)A \\
\le{}& \frac{r}{n+4} \times (n+4) = r.
\end{aligned}
$$

Hence, F maps B_r into B_r. Now, suppose that f satisfy the second condition. In this case, choose

$$\begin{aligned}
0 &< r \\
&\le \min\biggl\{\biggl(\frac{1}{(n+3)Ad}\biggr)^{\frac{1}{\rho-1}}, \biggl(\frac{1}{(n+3)Ad'\gamma_0^{p'}}\biggr)^{\frac{1}{\rho'-1}}, \biggl(\frac{1}{(n+3)Ad''\lambda_0^{p''}}\biggr)^{\frac{1}{\rho''-1}}, \\
&\quad \biggl(\frac{1}{(n+3)Ad_1}\biggr)^{\frac{1}{\rho_1-1}}, \biggl(\frac{1}{(n+3)Ad_2}\biggr)^{\frac{1}{\rho_2-1}}, \dots, \biggl(\frac{1}{(n+3)Ad_n}\biggr)^{\frac{1}{\rho_n-1}}\biggr\}.
\end{aligned}$$

By using similar arguments, one can show that $\|Fu\| \le \frac{r}{n+3} \times (n+3) = r$, and so F maps B_r into B_r. Since f is continuous, it is easy to get that F is also continuous. Now, we show that F is completely continuous operator on B_r. For each $u \in B_r$, put

$$N = \max_{t\in I}\bigl\{f\bigl(t, u(t), (\phi u)(t), (\psi u)(t), {}^cD^{\beta_1}u(t), {}^cD^{\beta_2}u(t), \dots, {}^cD^{\beta_n}u(t)\bigr)\bigr\} + 1.$$

For each $t_1, t_2 \in I$ with $t_1 < t_2$, we have

$$\begin{aligned}
&\bigl|(Fu)(t_2) - (Fu)(t_1)\bigr| \\
&\quad = \biggl|\int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, u(s), (\phi u)(s), (\psi u)(s), {}^cD^{\beta_1}u(s), {}^cD^{\beta_2}u(s), \dots, {}^cD^{\beta_n}u(s)\bigr)\,ds \\
&\qquad - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, u(s), (\phi u)(s), (\psi u)(s), {}^cD^{\beta_1}u(s), {}^cD^{\beta_2}u(s), \dots, {}^cD^{\beta_n}u(s)\bigr)\,ds \\
&\qquad + \frac{b(t_1-t_2)}{1+b}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\
&\qquad \times f\bigl(s, u(s), (\phi u)(s), (\psi u)(s), {}^cD^{\beta_1}u(s), {}^cD^{\beta_2}u(s), \dots, {}^cD^{\beta_n}u(s)\bigr)\,ds\biggr| \\
&\quad = \biggl|\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, u(s), (\phi u)(s), (\psi u)(s), {}^cD^{\beta_1}u(s), {}^cD^{\beta_2}u(s), \dots, {}^cD^{\beta_n}u(s)\bigr)\,ds \\
&\qquad + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, u(s), (\phi u)(s), (\psi u)(s), {}^cD^{\beta_1}u(s), {}^cD^{\beta_2}u(s), \dots, {}^cD^{\beta_n}u(s)\bigr)\,ds \\
&\qquad - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, u(s), (\phi u)(s), (\psi u)(s), {}^cD^{\beta_1}u(s), {}^cD^{\beta_2}u(s), \dots, {}^cD^{\beta_n}u(s)\bigr)\,ds \\
&\qquad + \frac{b(t_1-t_2)}{1+b}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\
&\qquad \times f\bigl(s, u(s), (\phi u)(s), (\psi u)(s), {}^cD^{\beta_1}u(s), {}^cD^{\beta_2}u(s), \dots, {}^cD^{\beta_n}u(s)\bigr)\,ds\biggr| \\
&\quad \le \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \\
&\qquad \times \bigl|f\bigl(s, u(s), (\phi u)(s), (\psi u)(s), {}^cD^{\beta_1}u(s), {}^cD^{\beta_2}u(s), \dots, {}^cD^{\beta_n}u(s)\bigr)\bigr|\,ds \\
&\qquad + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(s, u(s), (\phi u)(s), (\psi u)(s), {}^cD^{\beta_1}u(s), {}^cD^{\beta_2}u(s), \dots, {}^cD^{\beta_n}u(s)\bigr)\bigr|\,ds \\
&\qquad + \frac{|b|(t_2-t_1)}{|1+b|}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}
\end{aligned}$$

$$
\begin{aligned}
&\times \left|f\left(s,u(s),(\phi u)(s),(\psi u)(s),{}^cD^{\beta_1}u(s),{}^cD^{\beta_2}u(s),\dots,{}^cD^{\beta_n}u(s)\right)\right| ds \\
&\leq N\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds + N\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds \\
&\quad + \frac{N|b|(t_2-t_1)}{|1+b|}\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\,ds \\
&= \frac{N}{\Gamma(\alpha+1)}\left(t_2^\alpha - t_1^\alpha\right) + \frac{N|b|}{|1+b|\Gamma(\alpha)}(t_2-t_1).
\end{aligned}
$$

On the other hand, for each $i \in \{1,2,\dots,n\}$, we have

$$
\begin{aligned}
&\left|{}^cD^{\beta_i}(Fu)(t_2) - {}^cD^{\beta_i}(Fu)(t_1)\right| \\
&= \left|\int_0^{t_2}\frac{(t_2-s)^{-\beta_i}}{\Gamma(1-\beta_i)}(Fu)'(s)\,ds - \int_0^{t_1}\frac{(t_1-s)^{-\beta_i}}{\Gamma(1-\beta_i)}(Fu)'(s)\,ds\right| \\
&= \left|\int_0^{t_2}\frac{(t_2-s)^{-\beta_i}}{\Gamma(1-\beta_i)}\left(\int_0^s\frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\right.\right. \\
&\quad \times f\left(\tau,u(\tau),(\phi u)(\tau),(\psi u)(\tau),{}^cD^{\beta_1}u(\tau),{}^cD^{\beta_2}u(\tau),\dots,{}^cD^{\beta_n}u(\tau)\right) d\tau \\
&\quad - \frac{b}{1+b}\int_0^1\frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \\
&\quad \left.\times f\left(\tau,u(\tau),(\phi u)(\tau),(\psi u)(\tau),{}^cD^{\beta_1}u(\tau),{}^cD^{\beta_2}u(\tau),\dots,{}^cD^{\beta_n}u(\tau)\right) d\tau\right) ds \\
&\quad - \int_0^{t_1}\frac{(t_1-s)^{-\beta_i}}{\Gamma(1-\beta_i)}\left(\int_0^s\frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\right. \\
&\quad \times f\left(\tau,u(\tau),(\phi u)(\tau),(\psi u)(\tau),{}^cD^{\beta_1}u(\tau),{}^cD^{\beta_2}u(\tau),\dots,{}^cD^{\beta_n}u(\tau)\right) d\tau \\
&\quad - \frac{b}{1+b}\int_0^1\frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \\
&\quad \left.\left.\times f\left(\tau,u(\tau),(\phi u)(\tau),(\psi u)(\tau),{}^cD^{\beta_1}u(\tau),{}^cD^{\beta_2}u(\tau),\dots,{}^cD^{\beta_n}u(\tau)\right) d\tau\right) ds\right| \\
&= \left|\int_0^{t_1}\frac{(t_2-s)^{-\beta_i}}{\Gamma(1-\beta_i)}\left(\int_0^s\frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\right.\right. \\
&\quad \times f\left(\tau,u(\tau),(\phi u)(\tau),(\psi u)(\tau),{}^cD^{\beta_1}u(\tau),{}^cD^{\beta_2}u(\tau),\dots,{}^cD^{\beta_n}u(\tau)\right) d\tau \\
&\quad - \frac{b}{1+b}\int_0^1\frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \\
&\quad \left.\times f\left(\tau,u(\tau),(\phi u)(\tau),(\psi u)(\tau),{}^cD^{\beta_1}u(\tau),{}^cD^{\beta_2}u(\tau),\dots,{}^cD^{\beta_n}u(\tau)\right) d\tau\right) ds \\
&\quad + \int_{t_1}^{t_2}\frac{(t_2-s)^{-\beta_i}}{\Gamma(1-\beta_i)}\left(\int_0^s\frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\right. \\
&\quad \times f\left(\tau,u(\tau),(\phi u)(\tau),(\psi u)(\tau),{}^cD^{\beta_1}u(\tau),{}^cD^{\beta_2}u(\tau),\dots,{}^cD^{\beta_n}u(\tau)\right) d\tau \\
&\quad - \frac{b}{1+b}\int_0^1\frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \\
&\quad \left.\times f\left(\tau,u(\tau),(\phi u)(\tau),(\psi u)(\tau),{}^cD^{\beta_1}u(\tau),{}^cD^{\beta_2}u(\tau),\dots,{}^cD^{\beta_n}u(\tau)\right) d\tau\right) ds
\end{aligned}
$$

$$
\begin{aligned}
& - \int_0^{t_1} \frac{(t_1-s)^{-\beta_i}}{\Gamma(1-\beta_i)} \biggl(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \\
& \times f\bigl(\tau, u(\tau), (\phi u)(\tau), (\psi u)(\tau), {}^cD^{\beta_1}u(\tau), {}^cD^{\beta_2}u(\tau), \ldots, {}^cD^{\beta_n}u(\tau)\bigr)\, d\tau \\
& - \frac{b}{1+b} \int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \\
& \times f\bigl(\tau, u(\tau), (\phi u)(\tau), (\psi u)(\tau), {}^cD^{\beta_1}u(\tau), {}^cD^{\beta_2}u(\tau), \ldots, {}^cD^{\beta_n}u(\tau)\bigr)\, d\tau \biggr)\, ds \biggr| \\
= {} & \biggl| \int_0^{t_1} \frac{(t_2-s)^{-\beta_i} - (t_1-s)^{-\beta_i}}{\Gamma(1-\beta_i)} \\
& \times \biggl(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} f\bigl(\tau, u(\tau), (\phi u)(\tau), (\psi u)(\tau), \\
& {}^cD^{\beta_1}u(\tau), {}^cD^{\beta_2}u(\tau), \ldots, {}^cD^{\beta_n}u(\tau)\bigr)\, d\tau \\
& - \frac{b}{1+b} \int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \\
& \times f\bigl(\tau, u(\tau), (\phi u)(\tau), (\psi u)(\tau), {}^cD^{\beta_1}u(\tau), {}^cD^{\beta_2}u(\tau), \ldots, {}^cD^{\beta_n}u(\tau)\bigr)\, d\tau \biggr)\, ds \\
& + \int_{t_1}^{t_2} \frac{(t_2-s)^{-\beta_i}}{\Gamma(1-\beta_i)} \biggl(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \\
& \times f\bigl(\tau, u(\tau), (\phi u)(\tau), (\psi u)(\tau), {}^cD^{\beta_1}u(\tau), {}^cD^{\beta_2}u(\tau), \ldots, {}^cD^{\beta_n}u(\tau)\bigr)\, d\tau \\
& - \frac{b}{1+b} \int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \\
& \times f\bigl(\tau, u(\tau), (\phi u)(\tau), (\psi u)(\tau), {}^cD^{\beta_1}u(\tau), {}^cD^{\beta_2}u(\tau), \ldots, {}^cD^{\beta_n}u(\tau)\bigr)\, d\tau \biggr)\, ds \biggr| \\
\leq {} & \int_0^{t_1} \frac{(t_1-s)^{-\beta_i} - (t_2-s)^{-\beta_i}}{\Gamma(1-\beta_i)} \\
& \times \biggl(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \bigl| f\bigl(\tau, u(\tau), (\phi u)(\tau), (\psi u)(\tau), \\
& {}^cD^{\beta_1}u(\tau), {}^cD^{\beta_2}u(\tau), \ldots, {}^cD^{\beta_n}u(\tau)\bigr)\bigr|\, d\tau \biggr)\, ds \\
& + \frac{|b|}{|1+b|} \int_0^{t_1} \frac{(t_1-s)^{-\beta_i} - (t_2-s)^{-\beta_i}}{\Gamma(1-\beta_i)} \\
& \times \biggl(\int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \bigl| f\bigl(\tau, u(\tau), (\phi u)(\tau), (\psi u)(\tau), \\
& {}^cD^{\beta_1}u(\tau), {}^cD^{\beta_2}u(\tau), \ldots, {}^cD^{\beta_n}u(\tau)\bigr)\bigr|\, d\tau \biggr)\, ds \\
& + \int_{t_1}^{t_2} \frac{(t_2-s)^{-\beta_i}}{\Gamma(1-\beta_i)} \biggl(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \\
& \times \bigl| f\bigl(\tau, u(\tau), (\phi u)(\tau), (\psi u)(\tau), {}^cD^{\beta_1}u(\tau), {}^cD^{\beta_2}u(\tau), \ldots, {}^cD^{\beta_n}u(\tau)\bigr)\bigr|\, d\tau \biggr)\, ds \\
& + \frac{|b|}{|1+b|} \int_{t_1}^{t_2} \frac{(t_2-s)^{-\beta_i}}{\Gamma(1-\beta_i)} \biggl(\int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}
\end{aligned}
$$

$$\begin{aligned}
&\times \bigl|f\bigl(\tau,u(\tau),(\phi u)(\tau),(\psi u)(\tau),{}^cD^{\beta_1}u(\tau),{}^cD^{\beta_2}u(\tau),\dots,{}^cD^{\beta_n}u(\tau)\bigr)\bigr|\,d\tau\biggr)\,ds\\
&\le \frac{N}{\Gamma(\alpha)}\int_0^{t_1}\frac{(t_1-s)^{-\beta_i}-(t_2-s)^{-\beta_i}}{\Gamma(1-\beta_i)}s^{\alpha-1}\,ds\\
&\quad+\frac{N|b|}{|1+b|\Gamma(\alpha)}\int_0^{t_1}\frac{(t_1-s)^{-\beta_i}-(t_2-s)^{-\beta_i}}{\Gamma(1-\beta_i)}\,ds\\
&\quad+\frac{N}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{(t_2-s)^{-\beta_i}}{\Gamma(1-\beta_i)}s^{\alpha-1}\,ds+\frac{N|b|}{|1+b|\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{(t_2-s)^{-\beta_i}}{\Gamma(1-\beta_i)}\,ds\\
&\le \frac{(1+2|b|)N}{|1+b|\Gamma(\alpha)}\int_0^{t_1}\frac{(t_1-s)^{-\beta_i}-(t_2-s)^{-\beta_i}}{\Gamma(1-\beta_i)}\,ds\\
&\quad+\frac{(1+2|b|)N}{|1+b|\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{(t_2-s)^{-\beta_i}}{\Gamma(1-\beta_i)}\,ds\\
&\le \frac{(1+2|b|)N}{|1+b|\Gamma(\alpha)\Gamma(2-\beta_i)}\bigl[\bigl(t_2^{1-\beta_i}-t_1^{1-\beta_i}\bigr)+2(t_2-t_1)^{1-\beta_i}\bigr].
\end{aligned}$$

Hence,

$$\begin{aligned}
&\bigl\|Fu(t_2)-Fu(t_1)\bigr\|\\
&\quad\le \frac{N}{\Gamma(\alpha+1)}\bigl(t_2^{\alpha}-t_1^{\alpha}\bigr)+\frac{N|b|}{|1+b|\Gamma(\alpha)}(t_2-t_1)\\
&\qquad+\sum_{i=1}^{n}\frac{(1+2|b|)N}{|1+b|\Gamma(\alpha)\Gamma(2-\beta_i)}\bigl[\bigl(t_2^{1-\beta_i}-t_1^{1-\beta_i}\bigr)+2(t_2-t_1)^{1-\beta_i}\bigr],
\end{aligned}$$

which implies that $\|Fu(t_2)-Fu(t_1)\|\to 0$ as $t_1\to t_2$. Thus, F is equi-continuous and uniformly bounded. By using the Arzela-Ascoli theorem, one can get that F is completely continuous. Now, by using Theorem 2.5, F has a fixed point in B_r. Therefore, problem (1) has a solution. □

Corollary 2.7 *Let $f:[0,1]\times\mathbb{R}^{n+3}\to\mathbb{R}$ be a continuous function such that there exists a constant $l\in(0,\alpha-1)$ and a real-valued function $m(t)\in L^{\frac{1}{l}}([0,1],(0,\infty))$ such that $|f(t,x,y,w,u_1,u_2,\dots,u_n)|\le m(t)$ for all $t\in[0,1]$ and $x,y,w,u_1,u_2,\dots,u_n\in\mathbb{R}$. Then problem* (1) *has a solution.*

Finally, we should emphasize the importance of anti-periodic conditions, which appear in the case that $a=1$ and $b=1$ in the problem. For more details about this note, one can review the papers [24] and [25].

2.3 Examples

Example 2.1 Consider the boundary value problem

$$\begin{cases}
{}^cD^{\frac{7}{4}}u(t)=\frac{e^{-\pi t}}{24\sqrt{\pi}+e^{-\pi t}}\bigl[\frac{\sin t+e^t}{1+t^3}+\frac{|u(t)|}{1+|u(t)|}+\frac{e^{-\pi t}\cos\pi t}{1+t^2}\bigl(1+\frac{|(\phi u)(t)+{}^cD^{\frac{1}{2}}u(t)|}{1+|(\phi u)(t)+{}^cD^{\frac{1}{2}}u(t)|}\bigr)\\
\qquad\qquad+\frac{1+\sin^2\pi t}{2(t^{\frac{3}{2}}+4)}\bigl((\psi u)(t)+\frac{|{}^cD^{\frac{3}{4}}u(t)|}{1+|{}^cD^{\frac{3}{4}}u(t)|}\bigr)\bigr],\\
u(0)+u(1)=0,\qquad u'(0)+u'(1)=0,
\end{cases}\tag{2}$$

where $(\phi u)(t) = \int_0^t \frac{e^{-(s-t)}}{8} u(s)\,ds$ and $(\psi u)(t) = \int_0^t \frac{e^{-(s-t)/2}}{8} u(s)\,ds$ with $\gamma_0 = \frac{e-1}{8}$ and $\lambda_0 = \frac{\sqrt{e}-1}{4}$. Then, we have

$$\begin{aligned}
&\big|f\big(t,u(t),(\phi u)(t),(\psi u)(t),{}^cD^{\frac{1}{2}}u(t),{}^cD^{\frac{3}{4}}u(t)\big)\\
&\quad - f\big(t,v(t),(\phi v)(t),(\psi v)(t),{}^cD^{\frac{1}{2}}v(t),{}^cD^{\frac{3}{4}}v(t)\big)\big|\\
&\leq \frac{1}{24\sqrt{\pi}}\big(\big|u(t)-v(t)\big| + \big|(\phi u)(t)-(\phi v)(t)\big|\\
&\quad + \big|(\psi u)(t)-(\psi v)(t)\big| + \big|{}^cD^{\frac{1}{2}}u(t) - {}^cD^{\frac{1}{2}}v(t)\big|\\
&\quad + \big|{}^cD^{\frac{3}{4}}u(t) - {}^cD^{\frac{3}{4}}v(t)\big|\big).
\end{aligned}$$

Put $\mu(t) = \frac{1}{24\sqrt{\pi}} \in L^4([0,1],(0,\infty))$, $\kappa = \frac{1}{4}$ and $\mu^* = (\int_0^1 (\frac{1}{24\sqrt{\pi}})^4\,ds)^{\frac{1}{4}} = \frac{1}{24\sqrt{\pi}}$. Since $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$, $\Gamma(\frac{3}{4}) \approx 1/2{,}254$, $\Gamma(\frac{5}{4}) \approx 0/9{,}064$ and $\Gamma(\frac{7}{4}) \approx 0/9{,}191$, we have

$$\begin{aligned}
&(1+\gamma_0+\lambda_0)\biggl[\frac{3\mu^*}{2\Gamma(\alpha)}\left(\frac{1-\kappa}{\alpha-\kappa}\right)^{1-\kappa} + \frac{\mu^*}{4\Gamma(\alpha-1)}\left(\frac{1-\kappa}{\alpha-\kappa-1}\right)^{1-\kappa}\\
&\quad + \frac{\Gamma(\alpha-\kappa)\mu^*}{\Gamma(\alpha-1)}\left(\frac{1-\kappa}{\alpha-\kappa-1}\right)^{1-\kappa}\left(\frac{1}{\Gamma(\alpha-\beta_1-\kappa+1)} + \frac{1}{\Gamma(\alpha-\beta_2-\kappa+1)}\right)\\
&\quad + \frac{\mu^*}{2\Gamma(\alpha-1)}\left(\frac{1-\kappa}{\alpha-\kappa-1}\right)^{1-\kappa}\left(\frac{1}{\Gamma(2-\beta_1)} + \frac{1}{\Gamma(2-\beta_2)}\right)\biggr]\\
&= \left(1+\frac{e-1}{8}+\frac{\sqrt{e}-1}{4}\right)\biggl[\frac{1}{16\sqrt{\pi}\Gamma(\frac{7}{4})}\left(\frac{1}{2}\right)^{\frac{3}{4}} + \frac{1}{96\sqrt{\pi}\Gamma(\frac{3}{4})}\left(\frac{3}{2}\right)^{\frac{3}{4}}\\
&\quad + \frac{\Gamma(\frac{3}{2})}{24\sqrt{\pi}\Gamma(\frac{3}{4})}\left(\frac{3}{2}\right)^{\frac{3}{4}}\left(\frac{1}{\Gamma(2)} + \frac{1}{\Gamma(\frac{7}{4})}\right) + \frac{1}{48\sqrt{\pi}\Gamma(\frac{3}{4})}\left(\frac{3}{2}\right)^{\frac{3}{4}}\left(\frac{1}{\Gamma(\frac{3}{2})} + \frac{1}{\Gamma(\frac{5}{4})}\right)\biggr]\\
&\approx 0/1{,}466 < 1.
\end{aligned}$$

Thus, by using Theorem 2.3, the boundary value problem (2) has a unique solution.

Example 2.2 Consider the boundary value problem

$$\begin{cases}
{}^cD^\alpha u(t) = \frac{\lambda e^{-\pi t}}{\sqrt{1+t^2}} + \frac{\cos \pi t}{\sqrt{\pi + |u(t)| + |{}^cD^{\frac{1}{2}}u(t)|}}(u(t))^{\sigma_1} + \frac{e^{-\pi t}(1+\sin^2 u(t))}{(t+6)^3}((\phi u)(t))^{\sigma_2}\\
\qquad + \frac{tu(t)}{(4+t^2)(1+|u(t)|)}((\psi u)(t))^{\sigma_3}\\
\qquad + \frac{(1+\alpha)(t-\frac{1}{2})^2}{\Gamma(\alpha)(1+|u(t)+{}^cD^{\frac{3}{2}}u(t)|)}\sum_{k=1}^4 (\frac{\sin k\pi t}{2^k})({}^cD^{\beta_k}u(t))^{\delta_k},\\
u(0) + \frac{1}{2}u(1) = 0, \qquad u'(0) + \frac{2}{3}u'(1) = 0,
\end{cases} \tag{3}$$

where $\alpha = \frac{9}{5}$, $\beta_1 = \frac{1}{2}$, $\beta_2 = \frac{4}{5}$, $\beta_3 = \frac{2}{3}$, $\beta_4 = \frac{1}{7}$, $\lambda \in [0,\infty)$, $(\phi u)(t) = \int_0^t \frac{s^2 e^{-(s-t)/2}}{s^3+4} u(s)\,ds$ and $(\psi u)(t) = \int_0^t \frac{(t-s)^5}{\sqrt{1+s^2}} u(s)\,ds$. Then, we have

$$\begin{aligned}
&\big|f\big(t,u(t),(\phi u)(t),(\psi u)(t),{}^cD^{\beta_1}u(t),{}^cD^{\beta_2}u(t),{}^cD^{\beta_3}u(t),{}^cD^{\beta_4}u(t)\big)\big|\\
&\leq m(t) + \frac{1}{\sqrt{\pi}}\big|u(t)\big|^{\sigma_1} + \frac{1}{108}\big|(\phi u)(t)\big|^{\sigma_2} + \frac{1}{4}\big|(\psi u)(t)\big|^{\sigma_3} + \sum_{k=1}^4 \frac{(1+\alpha)}{\Gamma(\alpha)2^{k+2}}\big|{}^cD^{\beta_k}u(t)\big|^{\delta_k},
\end{aligned}$$

where $m(t) = \frac{\lambda e^{-\pi t}}{\sqrt{1+t^2}}$ for all $t \in [0,1]$. If $0 < \sigma_j < 1$, $0 < \delta_i < 1$ for $j = 1,2,3$ and $i = 1,2,3,4$, then assumption $(*)$ holds. If $\lambda = 0$ and $\sigma_j, \delta_i > 1$ for $j = 1,2,3$ and $i = 1,2,3,4$, then the second condition of Theorem 2.6 holds. Thus, by using Theorem 2.6, the boundary value problem (3) has a solution.

3 Conclusions

Fractional nonlinear differential equations, fractional integro-differential equations and their applications represent a topic of high interest in the area of fractional calculus and its applications in various fields of science and engineering [7]. In this article, based on the Schauder's fixed point theorem, we have proved some existence results for the multi-term nonlinear fractional integro-differential equation (1).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have equal contributions. All authors read and approved the final manuscript.

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